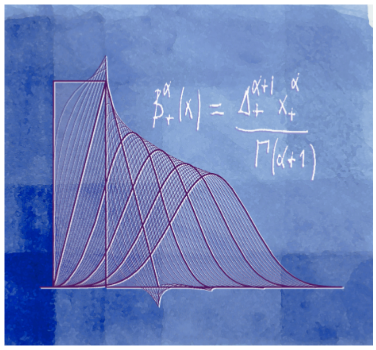
EPFL

Image Processing

Chapter 1
Characterization of continuous images

Prof. Michael Unser, LIB



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  - Multidimensional convolution
  - Modeling of optical systems
  - Examples of transfer functions

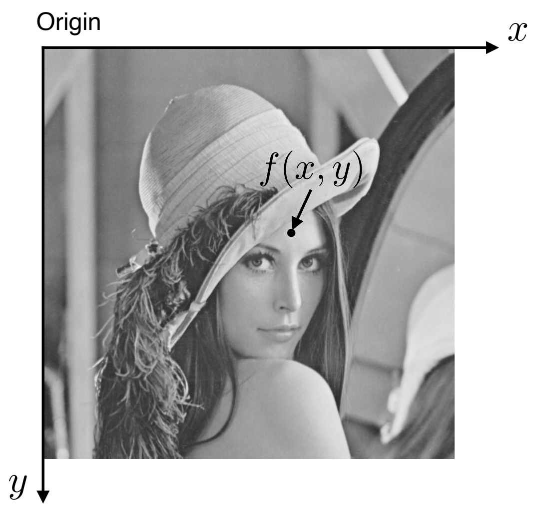
1.1 IMAGES AS FUNCTIONS

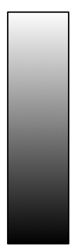
- Continuous image representation
- Hilbert-space formulation
- Space of finite-energy images
- Two-dimensional systems
- Linear, shift-invariant systems

Continuous image representation

2D light intensity function: $f(x,y)$

- spatial coordinates: (x,y)
- brightness (or gray level): $f \in [0, f_{\max}]$





Hilbert-space formulation

Hilbert space = infinite-dimensional Euclidean space

Unifying point of view: images as points in a Hilbert space $\mathcal{H}$

1D signals

- Vectors of samples $\qquad \mathbb{R}^N$

  $u = (u_1, u_2, \cdots, u_N)$
- Discrete signals $\qquad \ell_2(\mathbb{Z})$

  $u = (\cdots, u_0, u_1, \cdots, u_k, \cdots)$ or $u[\cdot]$
- Continuously-defined signals $\qquad L_2(\mathbb{R})$

  $u = u(x),\ x \in \mathbb{R}$ or $u(\cdot)$

2D images

- Finite arrays of pixels $\qquad \mathbb{R}^N$
- Discrete images $\qquad \ell_2(\mathbb{Z}^2)$
- Continuously-defined images $\qquad L_2(\mathbb{R}^2)$

Hilbert space: definition

A Hilbert space $\mathcal{H}$ is a *complete** vector space with an inner product.

*Completeness: all Cauchy sequences in $\mathcal{H}$ have a limit in $\mathcal{H}$.

- **$\mathcal{H}$-inner product**: $\langle u, v \rangle$

  (i) Linearity: $\langle a_1 u + a_2 v, w \rangle = a_1 \langle u, w \rangle + a_2 \langle v, w \rangle,$

  $\forall a_1, a_2 \in \mathbb{C},\ \forall u, v, w \in \mathcal{H}$

  (ii) Symmetry: $\langle u, v \rangle^* = \langle v, u \rangle, \qquad \forall u, v \in \mathcal{H}$

  (iii) Positive definite: $\langle u, u \rangle > 0, \qquad \forall u \neq 0, u \in \mathcal{H}$

- **Induced norm**

  $\|u\| = \langle u, u \rangle^{1/2}$

- **Cauchy-Schwarz inequality**

  $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$

Example: $u = (u_1, u_2, \cdots, u_N) \in \mathbb{C}^N \qquad \langle u, v \rangle = \sum_{n=1}^{N} u_n v_n^*$

Space of finite-energy images

- Images as 2D functions of the space variables

  $f(x,y) \in L_2(\mathbb{R}^2)$

  More compact vector notation: $f(\boldsymbol{x})$ with $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$

- 2D L_2-inner product

$$\langle f, g\rangle_{L_2} \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)g^*(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$\|f\|_{L_2} = \sqrt{\langle f, f\rangle_{L_2}} = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x,y)|^2\,\mathrm{d}x\mathrm{d}y}$$

- Space of finite-energy functions

$$L_2(\mathbb{R}^2) \triangleq \{f : \mathbb{R}^2 \to \mathbb{C} \quad \text{s.t.} \quad \|f\|_{L_2}^2 < +\infty\}$$

Space of finite energy images (cont'd)

- Extension to higher dimensions

  $f(\boldsymbol{x})$ with $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$

$$\langle f, g\rangle_{L_2(\mathbb{R}^d)} \triangleq \int_{\mathbb{R}^d} f(\boldsymbol{x})g^*(\boldsymbol{x})\,\mathrm{d}x_1\cdots\mathrm{d}x_d$$

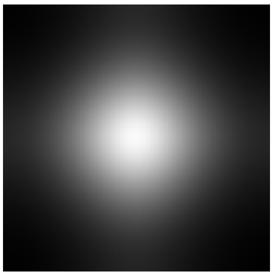
$$L_2(\mathbb{R}^d) \triangleq \{f : \mathbb{R}^d \to \mathbb{C} \quad \text{s.t.} \quad \|f\|_{L_2}^2 < +\infty\}$$

Examples of image functions

- 2D-Gaussian

$$g(x,y) = \frac{1}{2\pi}\exp\left(-\frac{(x^2+y^2)}{2}\right)$$

$$g(x,y) \in L_2(\mathbb{R}^2)$$



- Finite support Ω and bounded images

$$\begin{cases} \forall (x,y) \notin \Omega, & f(x,y) = 0 \\ \forall (x,y) \in \mathbb{R}^2, & |f(x,y)| < C_0 \end{cases} \implies \|f\|^2 < C_0^2 \cdot \int\int_\Omega \mathrm{d}x\mathrm{d}y$$

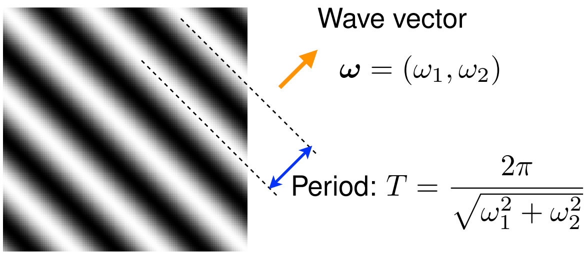
$$\Downarrow$$

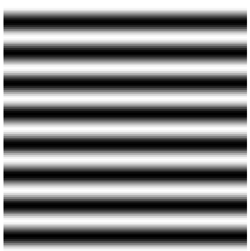
$$f \in L_2(\mathbb{R}^2)$$

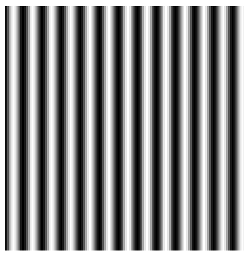
Plane waves

- Sinusoidal gratings

$$s(x,y) = A \cdot \cos(\omega_1 x + \omega_2 y + \phi) = A \cdot \cos(\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle + \phi)$$







Note that $s(x,y) \notin L_2(\mathbb{R}^2)$

However:

$$s(x,y) \cdot w(x,y) \in L_2(\mathbb{R}^2)$$

where

$w(x,y)$: finite-support and bounded window function.

Two-dimensional systems

- Mapping of an image function into another

$$\mathcal{T} : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}^2)$$

$$g(x, y) = \mathcal{T}\{f\}(x, y)$$

More (or less) pedantic notations:

$$g(\boldsymbol{x}) = \mathcal{T}\{f(\cdot)\}(\boldsymbol{x})$$

$$g = \mathcal{T}\{f\}$$

- Linear operators

$$\mathcal{T}\{a_1 f_1 + a_2 f_2\}(\boldsymbol{x}) = a_1 \mathcal{T}\{f_1\}(\boldsymbol{x}) + a_2 \mathcal{T}\{f_2\}(\boldsymbol{x})$$

$\forall f_1, f_2 \in \mathcal{H}$ and $\forall a_1, a_2 \in \mathbb{C}$

Examples

- Gradient operator is linear

$$\mathcal{T}_1\{f\} = f_x = \frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad \mathcal{T}_2\{f\} = f_y = \frac{\partial f(x, y)}{\partial y}$$

- Geometric operators are linear (warping)

$$\mathcal{T}_3\{f\}(x, y) = f\left(G_1(x, y), G_2(x, y)\right)$$

where $G_1(x, y)$ and $G_2(x, y)$ are arbitrary (non-linear)

- Threshold operator is non-linear

$$\mathcal{T}_4\{f\}(x, y) = \begin{cases} 1, & |f(x, y)| \geq T_0 \\ 0, & \text{otherwise} \end{cases}$$

Linear, shift-invariant systems

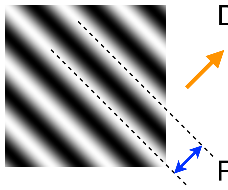
- **Definition.** $\mathcal{T}$ is shift-invariant iff: $\mathcal{T}\{f(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \mathcal{T}\{f(\cdot)\}(\boldsymbol{x} - \boldsymbol{x}_0)$

- Linear, shift-invariant system (LSI): model of most physical imaging devices

- **Complex sinusoids**: $s(x, y) = \exp\{j(\omega_x x + \omega_y y)\}$

Compact vector notation: $s(\boldsymbol{x}) = e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}$

with $\boldsymbol{\omega} = (\omega_x, \omega_y)$, $\boldsymbol{x} = (x, y)$



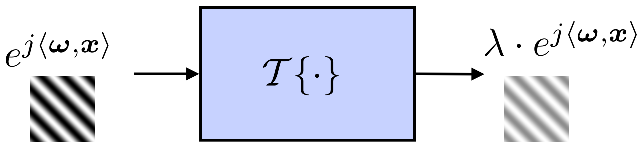
Direction of propagation: $\boldsymbol{u} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$

Period: $T = \frac{2\pi}{\|\boldsymbol{\omega}\|}$

Radial frequency: $\|\boldsymbol{\omega}\| = \sqrt{\omega_x^2 + \omega_y^2}$

> **Theorem.** The complex sinusoid $e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}$ is an *eigenfunction* of the LSI system $\mathcal{T}$ with eigenvalue $\lambda = \lambda(\boldsymbol{\omega}) = \mathcal{T}\{e^{j\langle \boldsymbol{\omega}, \cdot\rangle}\}(\mathbf{0})$.

Complex sinusoids and LSI systems



Proof: (in d dimensions)

- Input signal: $s(\boldsymbol{x}) = e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}$

- Shift $\Rightarrow$ Multiplication

$$s(\boldsymbol{x} - \boldsymbol{x}_0) = e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle - j\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle} = e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle} \cdot e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}$$

- If $\mathcal{T}$ is *linear* and *shift-invariant*

SI: $\mathcal{T}\{s(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \mathcal{T}\{s(\cdot)\}(\boldsymbol{x} - \boldsymbol{x}_0)$ for all $\boldsymbol{x}_0, \boldsymbol{x} \in \mathbb{R}^d$

Lin: $\mathcal{T}\{s(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \mathcal{T}\{e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle} \cdot e^{j\langle \boldsymbol{\omega}, \cdot\rangle}\}(\boldsymbol{x}) = e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle} \cdot \mathcal{T}\{s(\cdot)\}(\boldsymbol{x})$

Set $\boldsymbol{x}_0 = \boldsymbol{x}$: $\lambda = \mathcal{T}\{s\}(\mathbf{0}) = e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}\mathcal{T}\{s\}(\boldsymbol{x}) \Rightarrow \lambda \cdot e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} = \mathcal{T}\{s\}(\boldsymbol{x})$

1.2 MULTI-D FOURIER TRANSFORM

- Definition
- Separability
- Properties
- Dirac impulse
- Dirac related Fourier transforms
- Application: finding the orientation
- Importance of the phase

2D Fourier transform: definition

- 2D Fourier transform: $$\hat{f}(\omega_x, \omega_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{e}^{-\mathrm{j}(\omega_x x + \omega_y y)}\mathrm{d}x\mathrm{d}y$$
- Inverse Fourier transform: $$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{f}(\omega_x, \omega_y)\mathrm{e}^{\mathrm{j}(\omega_x x + \omega_y y)}\mathrm{d}\omega_x\mathrm{d}\omega_y$$

(a.e. = almost everywhere)

- Sufficient condition for existence:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x,y)|\,\mathrm{d}x\mathrm{d}y < +\infty \quad \Leftrightarrow \quad f \in L_1(\mathbb{R}^2)$$

Vector notation

Spatial variables: $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$

Frequency variables: $\boldsymbol{\omega} = (\omega_x, \omega_y) \in \mathbb{R}^2$

Equivalent phase: $\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle = \boldsymbol{\omega}^T\boldsymbol{x} = \omega_x x + \omega_y y$

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{j}\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\,\mathrm{d}x\mathrm{d}y$$

$$\Big\updownarrow \mathcal{F}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})\mathrm{e}^{\mathrm{j}\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\,\mathrm{d}\omega_x\mathrm{d}\omega_y$$

(a.e.)

Mathematical extensions

- Multidimensional Fourier transform (d dimensions)

Spatial variables: $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

Frequency variables: $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$

$$\hat{f}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega}) \triangleq \int_{\mathbb{R}^d} f(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle} \, \mathrm{d}x_1 \cdots \mathrm{d}x_d$$

$$\updownarrow \mathcal{F}$$

$$\mathcal{F}^{-1}\{\hat{f}\}(\boldsymbol{x}) \triangleq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) \mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle} \, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d = f(\boldsymbol{x})$$

(a.e.)

Sufficient condition for existence:

$f \in L_1(\mathbb{R}^d) \Rightarrow \hat{f}(\boldsymbol{\omega})$: bounded, continuous
and tends to 0 when $\|\boldsymbol{\omega}\| \rightarrow +\infty$

(Riemann-Lebesgue Lemma)

Finite-energy functions

- Fourier analysis in L_2 (Plancherel's extension)

$$f \in L_2(\mathbb{R}^d) \quad \Leftrightarrow \quad \hat{f} \in L_2(\mathbb{R}^d)$$

  - Parseval's formula: $\langle f, g \rangle_{L_2} \propto \langle \hat{f}, \hat{g} \rangle_{L_2}$

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) g^*(\boldsymbol{x}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_d = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) \hat{g}^*(\boldsymbol{\omega}) \, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$$

  - 2π-isometry (energy conservation)

$$\|f\|_{L_2}^2 = \frac{1}{(2\pi)^d} \|\hat{f}\|_{L_2}^2$$

Separability

- Separability of Fourier kernel: $e^{j(\omega_x x+\omega_y y)} = e^{j\omega_x x} \cdot e^{j\omega_y y}$

  Equivalent sequence of 1D calculations:

  (1) 1D Fourier transform along x ($y = \mathrm{Const}$)

  $$\hat{f}_y(\omega_x; y) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-j\omega_x x}\, dx$$

  (2) 1D Fourier transform along y ($x = \mathrm{Const}$)

  $$\hat{f}(\omega_x, \omega_y) = \int_{-\infty}^{+\infty} \hat{f}_y(\omega_x; y)\, e^{-j\omega_y y}\, dy$$

  Multidimensional Fourier transform inherits most properties of 1D Fourier transform!

- Separable signals (or transfer functions)

$$f(x,y) = f_1(x) \cdot f_2(y) \quad \Leftrightarrow \quad \hat{f}(\omega_x, \omega_y) = \hat{f}_1(\omega_x) \cdot \hat{f}_2(\omega_y)$$

In d dimensions: $$f(\boldsymbol{x}) = \prod_{i=1}^{d} f_i(x_i) \quad \Leftrightarrow \quad \hat{f}(\boldsymbol{\omega}) = \prod_{i=1}^{d} \hat{f}_i(\omega_i)$$

Fourier properties

- Duality: $\hat{f}(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} (2\pi)^d f(-\boldsymbol{\omega})$
- Symmetry: $f(\boldsymbol{x})$ real $\Leftrightarrow$ $\hat{f}^*(\boldsymbol{\omega}) = \hat{f}(-\boldsymbol{\omega})$
- Isometry: $\|f\| = (2\pi)^{-d/2} \|\hat{f}\|$
- Shift: $f(\boldsymbol{x} - \boldsymbol{x}_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0\rangle} \hat{f}(\boldsymbol{\omega})$
- Modulation: $e^{j\langle \boldsymbol{\omega}_0, \boldsymbol{x}\rangle} f(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
- Scaling: $f(\boldsymbol{x}/a) \overset{\mathcal{F}}{\longleftrightarrow} |a|^d \hat{f}(a\boldsymbol{\omega})$
- Affine transformation: $f(\boldsymbol{A}\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} |\det \boldsymbol{A}|^{-1} \hat{f}\left((\boldsymbol{A}^{-1})^T \boldsymbol{\omega}\right)$

  $\boldsymbol{A}$: non-singular matrix
- Differentiation: $\dfrac{\partial^n f(\boldsymbol{x})}{\partial x_i^n} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega_i)^n \hat{f}(\boldsymbol{\omega})$
- Moments: $$\mu_f^{m,n} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^m y^n f(x,y)\, dx dy = j^{m+n} \left. \frac{\partial^{m+n} \hat{f}(\boldsymbol{\omega})}{\partial \omega_x^m \partial \omega_y^n} \right|_{\boldsymbol{\omega}=\boldsymbol{0}}$$

  In particular: $\int_{\mathbb{R}^d} f(\boldsymbol{x})\, dx_1 \cdots dx_d = \hat{f}(\boldsymbol{0})$
- Convolution: $(f * g)(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\boldsymbol{\omega}) \cdot \hat{g}(\boldsymbol{\omega})$
- Multiplication: $f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{(2\pi)^d} (\hat{f} * \hat{g})(\boldsymbol{\omega})$

Example of computation

- 2D Gaussian

$$g(x,y) = e^{-(x^2+y^2)/2}$$

1. Use separability

$$g(x,y) = e^{-x^2/2} \cdot e^{-y^2/2} \quad \Rightarrow \quad \hat{g}(\omega_x,\omega_y) = \hat{f}(\omega_x)\cdot\hat{f}(\omega_y)$$

where $f(x) = e^{-x^2/2} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-j\omega x} \mathrm{d}x$

2. Determine 1D Fourier transform

Table or explicit calculation

$$e^{-x^2/2} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \sqrt{2\pi}\, e^{-\omega^2/2}$$

$$\Rightarrow \quad \hat{g}(\omega_x,\omega_y) = 2\pi \cdot e^{-(\omega_x^2+\omega_y^2)/2}$$

Generalized Parseval relation

- Let f and φ be two d-dimensional images with Fourier transform $\hat{f} = \mathcal{F}\{f\}$ and $\hat{\varphi} = \mathcal{F}\{\varphi\}$ and such that the inner product $\langle f, g\rangle$ (resp. duality product) is well defined. Then

(1) $\quad \langle f,\varphi\rangle = \frac{1}{(2\pi)^d}\langle \hat{f},\hat{\varphi}\rangle \quad$ (generalized Parseval relation)

- Primary conditions of validity:
  (i) $(f,\varphi) \in L_2(\mathbb{R}^d) \times L_2(\mathbb{R}^d)$ (Plancherel)
  (ii) $(f,\varphi) \in L_\infty(\mathbb{R}^d) \times L_1(\mathbb{R}^d)$
  (iii) $(f,\varphi) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ (Schwartz)

- Used to define the Fourier transform when the classical Fourier integral is undefined.

- Generalized Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$

$\hat{f} = \mathcal{F}\{f\} \in \mathcal{S}'(\mathbb{R}^d)$ is the (generalized) Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$

$\Leftrightarrow \quad \langle f,\varphi\rangle = \frac{1}{(2\pi)^d}\langle \mathcal{F}\{f\},\hat{\varphi}\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$

where $\hat{\varphi} = \mathcal{F}\{\varphi\} = \int_{\mathbb{R}^d} \varphi(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\langle\cdot,\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x} \in \mathcal{S}(\mathbb{R}^d)$

Dirac impulse

Abstract definition: $\forall f \in C^0(\mathbb{R}), \quad \langle f, \delta \rangle = \int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x = f(0)$

$C^0(\mathbb{R})$: the space of continuous functions over $\mathbb{R}$

- Properties
  - Normalized integral: $\int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = 1$ $\quad (f(x) = 1)$
  - Fourier transform: $\delta(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \int_{-\infty}^{+\infty} \delta(x)e^{-j\omega x}\mathrm{d}x = 1$ $\quad (f(x) = e^{-j\omega x})$
  - Convolution: $\forall g \in C^0, \quad (g * \delta)(x) = g(x)$ $\quad (f(\cdot) = g(x - \cdot))$

- Explicit construction
  - Window function $\varphi(x) \in L_1(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} \varphi(x)\mathrm{d}x = 1$

    e.g., $\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$
  - Integral-preserving dilation/contraction: $\int_{-\infty}^{+\infty} \frac{1}{|a|}\varphi\left(\frac{x}{a}\right) \mathrm{d}x = 1$
  - $\delta(x) = \lim_{a \to 0} \left(\frac{1}{|a|}\varphi\left(\frac{x}{a}\right)\right)$

Multidimensional Dirac impulse

Abstract definition: $\forall f \in C^0(\mathbb{R}^d), \quad \langle f, \delta \rangle = \int_{\mathbb{R}^d} f(\boldsymbol{x})\delta(\boldsymbol{x})\, \mathrm{d}x_1 \cdots \mathrm{d}x_d = f(\mathbf{0})$

Multidimensional Dirac impulse is separable:

$$\delta(x, y) = \delta(x) \cdot \delta(y) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad 1$$

In d dimensions: $\qquad \delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$

- Properties
  - Normalized integral: $\langle \delta, 1 \rangle = \int_{\mathbb{R}^d} \delta(\boldsymbol{x})\, \mathrm{d}x_1 \cdots \mathrm{d}x_d = 1$
  - Fourier transform: $\delta(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \int_{\mathbb{R}^d} \delta(\boldsymbol{x})e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}\, \mathrm{d}x_1 \cdots \mathrm{d}x_d = 1$
  - Multiplication: $\forall f \in C^0, \quad f(\boldsymbol{x}) \cdot \delta(\boldsymbol{x} - \boldsymbol{x}_0) = f(\boldsymbol{x}_0)\delta(\boldsymbol{x} - \boldsymbol{x}_0)$
  - Sampling: $\forall f \in C^0, \quad \langle f, \delta(\cdot - \boldsymbol{x}_0)\rangle = \int_{\mathbb{R}^d} f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_0)\, \mathrm{d}x_1 \cdots \mathrm{d}x_d = f(\boldsymbol{x}_0)$
  - Convolution: $\forall f \in C^0, \quad (f * \delta)(\boldsymbol{x}) = f(\boldsymbol{x})$
  - Scaling: $\delta(\boldsymbol{x}/a) = |a|^d \cdot \delta(\boldsymbol{x})$

Dirac-related Fourier transforms

- Constant

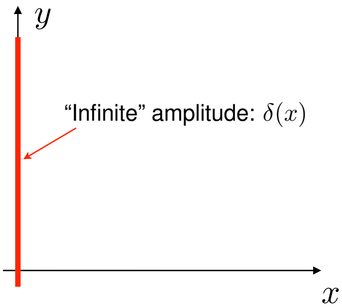
One-dimensional: $1 \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{+\infty} e^{-j\omega x} \mathrm{d}x = ?$

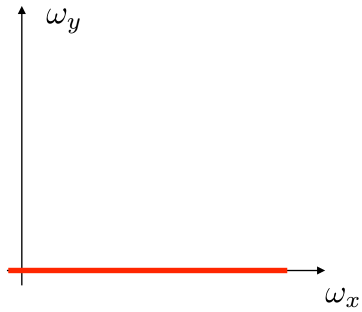
$$\lim_{A \to +\infty} \int_{-A}^{+A} e^{-j\omega x} \mathrm{d}x = 2\pi \cdot \delta(\omega)$$ (or by duality)

Multidimensional: $1 \stackrel{\mathcal{F}}{\longleftrightarrow} (2\pi)^d \, \delta(\boldsymbol{\omega})$

- Ideal line

$$f(x,y) = \delta(x) \cdot 1 = f_1(x) \cdot f_2(y) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{f}_1(\omega_x) \cdot \hat{f}_2(\omega_y) = 1 \cdot 2\pi\delta(\omega_y)$$

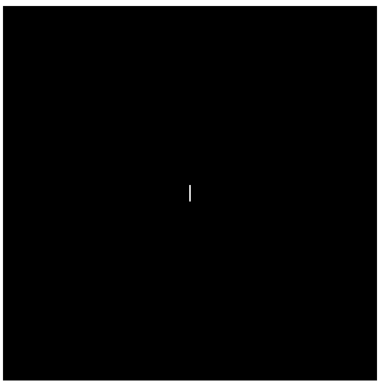


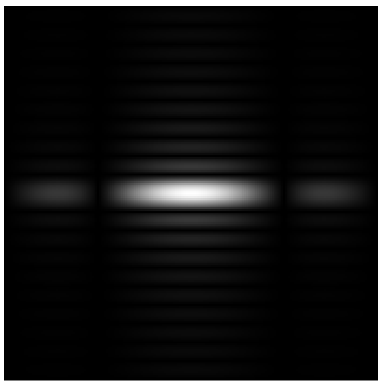


More realistic line model

- Rectangular shape

$$f(x,y) = \mathrm{rect}(x/a) \cdot \mathrm{rect}(y/A) \stackrel{\mathcal{F}}{\longleftrightarrow} |a| \,\mathrm{sinc}\left(\frac{a\,\omega_x}{2\pi}\right) \cdot |A| \,\mathrm{sinc}\left(\frac{A\,\omega_y}{2\pi}\right)$$





Reminder:

$$\mathrm{rect}(x) = \begin{cases} 1, & x \in [-\frac{1}{2}, +\frac{1}{2}] \\ 0, & \text{otherwise} \end{cases} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$

Application: finding the orientation

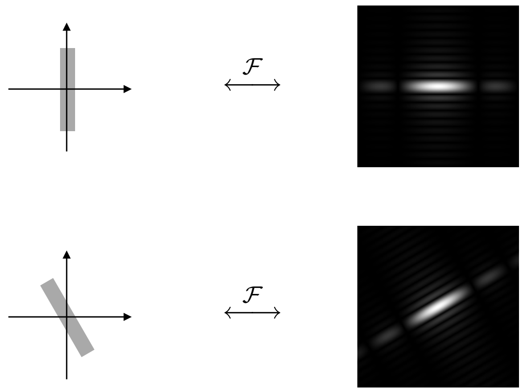
Problem: Design a (real time?) system that can determine the orientation of a linear pattern placed at an arbitrary location in the image.

- Reasons for working in the Fourier domain
  - Translation invariance

$$g(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{x}_0) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}(\boldsymbol{\omega}) \cdot e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0 \rangle} \qquad \Rightarrow \qquad |\hat{g}(\boldsymbol{\omega})| = |\hat{f}(\boldsymbol{\omega})|$$

  - Rotation property

$$g_\theta(\boldsymbol{x}) = f(\boldsymbol{R}_\theta \boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}(\boldsymbol{R}_\theta \boldsymbol{\omega})$$



Problem solution

- Compute Fourier inertia matrix

$$\boldsymbol{M} = \begin{bmatrix} \int\int \omega_x^2 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_x \mathrm{d}\omega_y & \int\int \omega_x \omega_y |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_x \mathrm{d}\omega_y \\ \int\int \omega_y \omega_x |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_x \mathrm{d}\omega_y & \int\int \omega_y^2 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_x \mathrm{d}\omega_y \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \langle j\omega_x \hat{f}(\boldsymbol{\omega}), \, j\omega_x \hat{f}(\boldsymbol{\omega}) \rangle & \langle j\omega_x \hat{f}(\boldsymbol{\omega}), \, j\omega_y \hat{f}(\boldsymbol{\omega}) \rangle \\ \langle j\omega_y \hat{f}(\boldsymbol{\omega}), \, j\omega_x \hat{f}(\boldsymbol{\omega}) \rangle & \langle j\omega_y \hat{f}(\boldsymbol{\omega}), \, j\omega_y \hat{f}(\boldsymbol{\omega}) \rangle \end{bmatrix}$$

- Determine axes of inertia

Eigen-decomposition: $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$

$\boldsymbol{u}_1$: eigenvector in the direction of the long axis

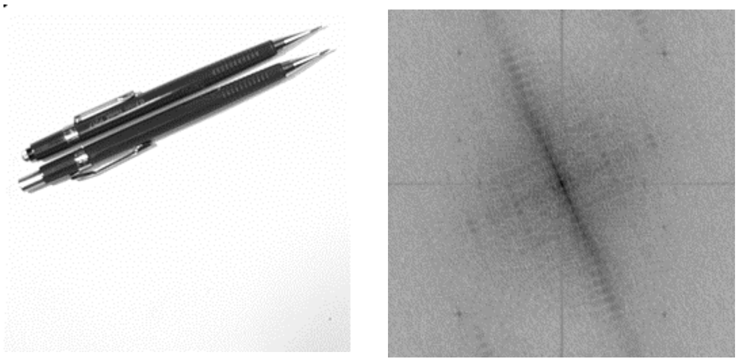
$\boldsymbol{u}_2$: eigenvector in the direction of the short axis

- Fast algorithm: $\boldsymbol{M} = \begin{bmatrix} \quad \end{bmatrix}$

Orientation estimation: examples

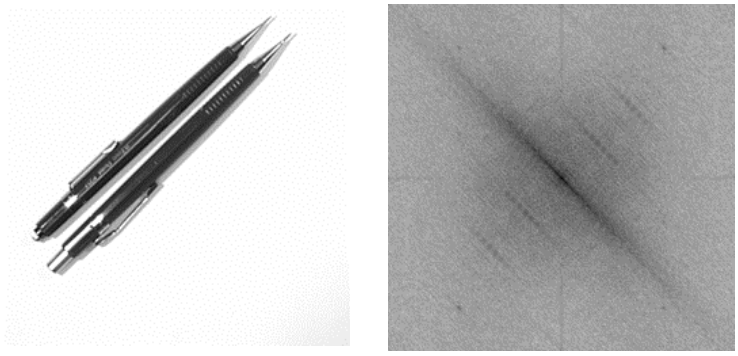
- Image 1:

  Measured angle: 25° ± 2°
  Computed angle: 27°



- Image 2

  Measured angle: 44° ± 2°
  Computed angle: 45.6°



Importance of the phase

- Fourier transform

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle} \, \mathrm{d}x_1 \cdots \mathrm{d}x_d = \left| \hat{f}(\boldsymbol{\omega}) \right| \cdot \exp\left(j \Phi_{\hat{f}}(\boldsymbol{\omega}) \right)$$

- Fourier modulus:

$$\left| \hat{f}(\boldsymbol{\omega}) \right| = \left(\hat{f}(\boldsymbol{\omega}) \cdot \hat{f}^*(\boldsymbol{\omega}) \right)^{1/2} = \sqrt{\mathrm{Re}\left[\hat{f}(\boldsymbol{\omega}) \right]^2 + \mathrm{Im}\left[\hat{f}(\boldsymbol{\omega}) \right]^2}$$

- Fourier phase:

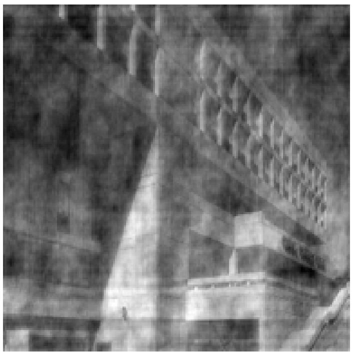
$$\Phi_{\hat{f}}(\boldsymbol{\omega}) = \arg\left(\hat{f}(\boldsymbol{\omega}) \right) = \arctan\left(\frac{\mathrm{Im}\left[\hat{f}(\boldsymbol{\omega}) \right]}{\mathrm{Re}\left[\hat{f}(\boldsymbol{\omega}) \right]} \right)$$

Module or phase ?

Image 1

Image 2

Module(Image2), Phase(Image1)



Module(Image1), Phase(Image2)

1.3 CHARACTERIZATION OF LSI

Linearity: $\mathcal{T}\{a_1 f_1 + a_2 f_2\}(\boldsymbol{x}) = a_1 \mathcal{T}\{f_1\}(\boldsymbol{x}) + a_2 \mathcal{T}\{f_2\}(\boldsymbol{x})$

Shift-invariance: $\mathcal{T}\{f(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \mathcal{T}\{f(\cdot)\}(\boldsymbol{x} - \boldsymbol{x}_0)$

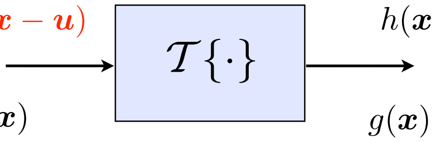
- Multidimensional convolution
- 2D convolution theorem
- Modeling of optical systems
- Examples of transfer functions

LSI system as a convolution operator

Image as a "sum" of Dirac impulses: $f(\boldsymbol{x}) = (\delta * f)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})\delta(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}u_1\cdots\mathrm{d}u_d$

- Response of a linear system (superposition principle)

$\delta(\boldsymbol{x}-\boldsymbol{u})$ → $h(\boldsymbol{x},\boldsymbol{u})$ or $h_{\mathrm{LSI}}(\boldsymbol{x}-\boldsymbol{u})$



$f(\boldsymbol{x})$ → $g(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})h(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}u_1\cdots\mathrm{d}u_d$

  - Impulse response (possibly, space-dependent) : $h(\boldsymbol{x},\boldsymbol{u}) = \mathcal{T}\{\delta(\cdot-\boldsymbol{u})\}(\boldsymbol{x})$
  - Arbitrary input: $\mathcal{T}\{f\}(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})\,\mathcal{T}\{\delta(\cdot-\boldsymbol{u})\}(\boldsymbol{x})\,\mathrm{d}u_1\cdots\mathrm{d}u_d$ (by linearity)

- Linear, shift-invariant system
  - Impulse response (or point-spread function): $h(\boldsymbol{x}) = \mathcal{T}_{\mathrm{LSI}}\{\delta(\cdot)\}(\boldsymbol{x}) = h(\boldsymbol{x},\boldsymbol{0})$
  - Shift-invariance $\Rightarrow$ $\mathcal{T}_{\mathrm{LSI}}\{\delta(\cdot-\boldsymbol{u})\}(\boldsymbol{x}) = h(\boldsymbol{x}-\boldsymbol{u})$
  - Arbitrary input: $\mathcal{T}_{\mathrm{LSI}}\{f\}(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}u_1\cdots\mathrm{d}u_d = (h*f)(\boldsymbol{x})$

2D Convolution theorem

- 2D convolution integral

$$\begin{aligned}(f*h)(x,y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u,v)h(x-u,y-v)\mathrm{d}u\mathrm{d}v \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u,v)f(x-u,y-v)\mathrm{d}u\mathrm{d}v = (h*f)(x,y)\end{aligned}$$

- Convolution theorem: $(f*h)(\boldsymbol{x}) \xleftrightarrow{\mathcal{F}} \hat{f}(\boldsymbol{\omega})\hat{h}(\boldsymbol{\omega})$

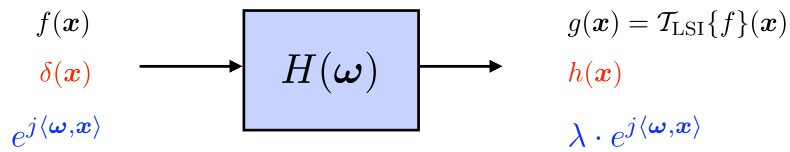
Proof: (in d dimensions) $g(\boldsymbol{x}) = (f*h)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}u_1\cdots\mathrm{d}u_d$

$$\updownarrow \mathcal{F}$$

$$\begin{aligned}\hat{g}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}u_1\cdots\mathrm{d}u_d\right)e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\,\mathrm{d}x_1\cdots\mathrm{d}x_d \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(\boldsymbol{u})h(\boldsymbol{v})e^{-j\langle\boldsymbol{\omega},\boldsymbol{u}+\boldsymbol{v}\rangle}\,\mathrm{d}u_1\cdots\mathrm{d}u_d\right)\mathrm{d}v_1\cdots\mathrm{d}v_d \quad \text{(change of variable } \boldsymbol{v}=\boldsymbol{x}-\boldsymbol{u}\text{)} \\ &= \int_{\mathbb{R}^d} f(\boldsymbol{u})e^{-j\langle\boldsymbol{\omega},\boldsymbol{u}\rangle}\,\mathrm{d}u_1\cdots\mathrm{d}u_d \int_{\mathbb{R}^d} h(\boldsymbol{v})e^{-j\langle\boldsymbol{\omega},\boldsymbol{v}\rangle}\,\mathrm{d}v_1\cdots\mathrm{d}v_d\end{aligned}$$

Technical hypothesis: $h, f \in L_1(\mathbb{R}^d) \Rightarrow g \in L_1(\mathbb{R}^d)$

Transfer function



linear shift-invariant system

- Method 1: identify the impulse response

$H(\boldsymbol{\omega})$: Engineer notation for $\hat{h}(\boldsymbol{\omega})$

$$g(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) h(\boldsymbol{x}-\boldsymbol{y})\, \mathrm{d}y_1 \cdots \mathrm{d}y_d = (h * f)(\boldsymbol{x})$$

Transfer function: $\mathcal{F}\{h\} \quad \Rightarrow \quad H(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} h(\boldsymbol{x}) e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\, \mathrm{d}x_1 \cdots \mathrm{d}x_d$

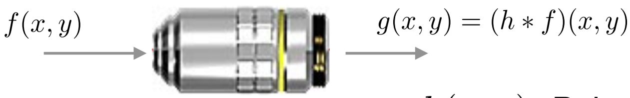
- Method 2: Fourier-domain formulation

Input/output relation: $\hat{g}(\boldsymbol{\omega}) = H(\boldsymbol{\omega}) \cdot \hat{f}(\boldsymbol{\omega}) \quad \Rightarrow \quad$ Transfer function: $H(\boldsymbol{\omega}) = \dfrac{\hat{g}(\boldsymbol{\omega})}{\hat{f}(\boldsymbol{\omega})}$

- Method 3: use eigenfunction property of complex sinusoids

Compute: $\mathcal{T}_{\mathrm{LSI}}\{e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\} = \lambda \cdot e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} \quad \Rightarrow \quad H(\boldsymbol{\omega}) = \lambda = \dfrac{\mathcal{T}_{\mathrm{LSI}}\{e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\}}{e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}}$

Modeling of optical systems



$h(x,y)$: Point Spread Function (PSF)

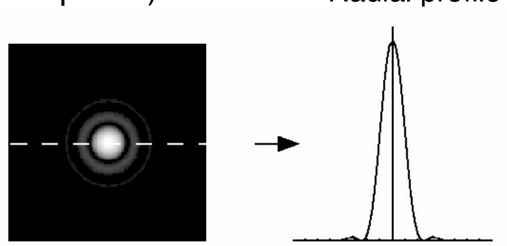
Diffraction-limited optics = LSI system

- Aberration-free point spread function (in focal plane)

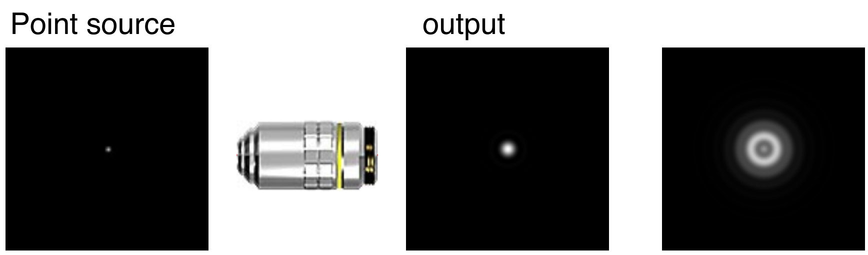
$$h(x,y) = h(r) = C \cdot \left[\frac{2J_1(\pi r)}{\pi r}\right]^2$$

where $r = \sqrt{x^2 + y^2}$ (radial distance)

Radial profile



- Effect of misfocus

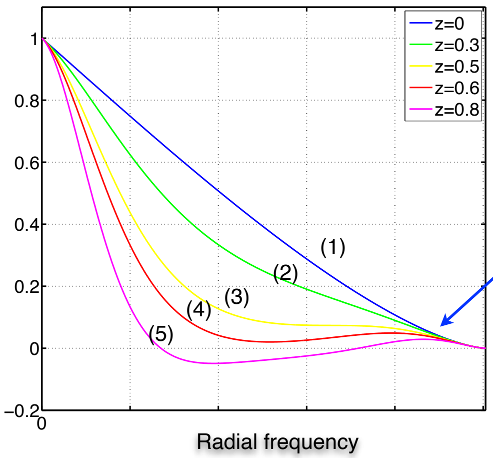


(in focus) $z = 0$ (defocus)

Optical transfer function

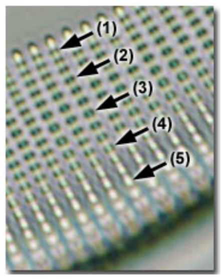
- $H(\omega_x, \omega_y) = \mathcal{F}\{h\}$: Optical transfer function (OTF)
- $|H(\omega_x, \omega_y)|$: Modulation transfer function (MTF)

- Isotropic: $H(\omega_x, \omega_y) = H(\omega)$ where $\omega = \sqrt{\omega_x^2 + \omega_y^2}$ (radial frequency)



NA: numerical aperture of the lense

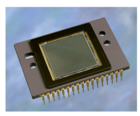
$$\omega_c = \frac{\mathrm{NA}}{\pi \cdot \lambda_{\text{light}}}$$



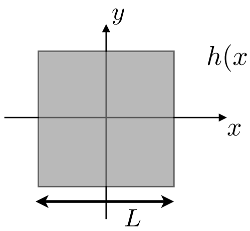
Transfer function of a lens for various degrees of misfocus

Examples of transfer functions

- CCD camera



Sampling aperture (photosite or "pixel" integration area):



$$h(x,y) = \frac{1}{L^2}\text{rect}\left(\frac{x}{L}\right)\cdot\text{rect}\left(\frac{y}{L}\right) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(\omega_x, \omega_y) = \text{sinc}\left(\frac{L\omega_x}{2\pi}\right)\cdot\text{sinc}\left(\frac{L\omega_y}{2\pi}\right)$$

- Motion blur

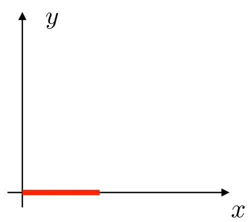
Hypothesis: translational motion of the camera: $\boldsymbol{x}_0(t)$

$$g(\boldsymbol{x}) = \frac{1}{T}\int_0^T f(\boldsymbol{x} - \boldsymbol{x}_0(t))\mathrm{d}t \quad \Rightarrow \quad H(\boldsymbol{\omega}) = \frac{1}{T}\int_0^T e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0(t)\rangle}\mathrm{d}t$$

Example:

uniform motion in x: $\boldsymbol{x}_0(t) = (at/T, 0)$

$$H(\boldsymbol{\omega}) = e^{-ja\omega_x/2}\text{sinc}\left(\frac{a\omega_x}{2\pi}\right) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad h(x,y) = \frac{1}{|a|}\text{rect}\left(\frac{x - a/2}{a}\right)\cdot\delta(y)$$



1.4 SUMMARY

- Continuous-space images are modeled as functions $f(x, y)$ of the spatial variables x and y.
- These functions have finite energy: $f \in L_2$. It is convenient to view them as points in a Hilbert space.
- A continuous-domain image-processing operator is a mapping $\mathcal{T} : L_2 \to L_2$.
- The complex sinusoids $e^{j\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle}$ are the eigenfunctions of linear shift-invariant (LSI) systems. They are $(2\pi/\|\boldsymbol{\omega}\|)$-periodic plane waves that propagate in the direction $\boldsymbol{\omega}$.
- The 2D Fourier transform of an image reveals its spatial frequency content. The Fourier phase contains the information most relevant perceptually (contours).
- The 2D Fourier transform is very similar to the 1D one; one simply replaces the scalar variables x and ω by vectors. Thus, it has essentially the same properties.
- The 2D Fourier transform of a separable signal $f(x, y) = f_1(x) f_2(y)$ should be determined using 1D transforms only.
- A LSI system performs a convolution.
- Continuous-space LSI systems are entirely characterized by their impulse response (point-spread function or sampling aperture), $h(x, y) = \mathcal{T}_{\mathrm{LSI}}\{\delta\}(x, y)$, or, equivalently, by their transfer function $H(\omega_x, \omega_y) = \mathcal{F}\{h\}(\omega_x, \omega_y)$.